$$\frac{d\lambda}{df} = -\frac{\lambda}{f}, \quad (43)$$

then

$$\frac{dz_i}{df} = \frac{z_n}{2}\sec^2\theta/2\cdot\frac{\theta}{f}\cdot\left(\frac{\lambda g}{\lambda}\right)^2. \quad (44)$$

When $f=f_0$ (6200 mc),

$$\left(\frac{\lambda g}{\lambda}\right)^2 \approx 2 \quad \text{and} \quad \theta = \frac{\pi}{2},$$

$$\left(\frac{dz_i}{df}\right)_{f=f_0} = \frac{\pi z_n}{f}. \quad (45)$$

Thus, to a first-order approximation valid over a small frequency range around f_0, the slope of z_i is linear and positive with frequency, and hence the short-circuit stub can be represented as an inductance L_n, in conjunction with a certain frequency variable, ω', given by

$$L_n = \frac{z_n}{\omega} \text{ (henries, with respect to unity impedance level)}, \quad (46)$$

$$\omega' = \omega_0\left(1 + \pi\frac{\omega - \omega_0}{\omega_0}\right). \quad (47)$$

A similar argument applies to the open-circuit stubs, the slope of the input admittance being linear and positive with frequency to a first-order approximation so that the open-circuit stub can be represented as a capacity C_n, in conjunction with the frequency transformation of (47). C_n is given by

$$C_n = \frac{y_n}{\omega_0} \text{ (farads, with respect to unity impedance level)}. \quad (48)$$

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The Analogy between the Weissfloch Transformer and the Ideal Attenuator (Reflection Coefficient Transformer) and an Extension to Include the General Lossy Two-Port*

Weissfloch's transformer theorem states that at certain pairs of reference planes a lossless two-port can be represented by an ideal transformer. There are many proofs of this important theorem. One of the most interesting is due to Bolinder,[1] who uses properties of the bilinear transformation. For lossless two-ports, the transformations will belong to the Fuchsian[2] group. This means that the isometric[1,2] circles are orthogonal to the principal circle. In the reflection coefficient plane (where Bolinder proves the theorem), the principal circle is the unit circle. The fixed points of the transformation will be on the unit circle or in a pair inverse with respect to the unit circle. Bolinder then uses lengths of lossless line to move the fixed points to the positions $\Gamma = \pm 1$. In the impedance plane this corresponds to fixed points of 0 and ∞. Therefore the transformation can be written as $Z' = k^2 Z$ and the transformer theorem is proven. The transformation through the two-port at any pair of reference planes, in either the reflection coefficient or the impedance plane, can be done by inversion in the isometric circles and a reflection in the line of symmetry,[3] as described by Ford[2] and Bolinder.[1]

The reflection coefficient transformer-(ideal attenuator), described by Altschulter and Kahn,[4] has a scattering matrix

$$S = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}$$

where K is a real number and, for an attenuator, less than unity. The transformation in the reflection coefficient plane is $\Gamma' = K^2\Gamma$, while in the impedance plane the corresponding relation is

$$Z' = \frac{\dfrac{Z(1+K^2)}{2K} + \dfrac{(1-K^2)}{2K}}{\dfrac{Z(1-K^2)}{2K} + \dfrac{1+K^2}{2K}}.$$

The fixed points of this transformation are ± 1 in the impedance plane and 0 and ∞ in the reflection coefficient plane. Both transformers produce hyperbolic transfor-

* Received by the PGMTT, February, 1959; revised, March, 1959.

[1] E. F. Bolinder, "Impedance and polarization-ratio transformations by a graphical method using the isometric circles," IRE Trans. on Microwave Theory and Techniques, vol. MTT-4, pp. 176–180; July, 1956.

[2] L. R. Ford, "Automorphic Functions," 2nd ed., Chelsea Publishing Co., New York, N. Y.; 1951.

[3] If the transformation is loxodromic, a rotation must be added.

[4] H. M. Altschulter and W. K. Kahn, "Nonreciprocal two-ports represented by modified Wheeler networks," IRE Trans. on Microwave Theory and Techniques, vol. MTT-4, pp. 228–233; October, 1956.

mations, since the trace of the normalized transformation is real and its magnitude is greater than two.[5] At the proper reference planes (where the transformations are the same), problems involving one of the two-ports *per se* in the impedance plane can be solved considering the other two-port in the reflection coefficient plane. It should be pointed out that analogy will fail when the reference planes are moved, because for the loss-less two-port the transformation is always nonloxodromic, while for the ideal attenuator the transformation will become, in general, loxodromic. It is also of interest to note that the fixed points of the reflection coefficient transformer are independent of the choice of reference planes.

In the general case of a lossy two-port, it is shown in the Appendix that pairs of reference planes can be found where the transformation is nonloxodromic. At these reference terminals the lossy two-port will be analogous to the lossless two-port which has the same fixed points in the reflection coefficient plane and which has the same multipliers of the transformations in canonical[2] form. This is true for all hyperbolic transformations where the two fixed points (the iterative impedances) have the same magnitude, all parabolic transformations (which have only one fixed point), and all elliptic transformations where the phase angles of the fixed points are equal. This class of transformation includes all symmetric networks with positive resistive components. Therefore, for this special class of lossy networks (eacj at its proper pair of reference planes) the transformations belong to the Fuchsian group. Therefore, any of the several lossy networks can be "replaced" (in the sense that it has the same transformation) by an "analogous" lossless network. If the fixed points in the impedance plane for a hyperbolic transformation do not have equal magnitude, then a transformation is used to map the fixed points in the impedance plane into fixed points in the new W plane with magnitude equal to unity and for convenience equal to ± 1. In this W plane, the analogy with the lossless two-port in the Γ plane holds. Similarly, for the elliptic transformation, the fixed points in the impedance plane are moved to points inverse with the unit circle and an "analogous" elliptic lossless network can be found. Therefore, it has been shown that any lossy two-port can be "replaced" by an "analogous" lossless network.

An Alternate Viewpoint

If the constructions for the isometric circle method becomes unwieldy because of the unbounded nature of Euclidean space, the impedance and/or the reflection coefficient planes can be transformed into the Cayley-Klein diagram (projective chart), which has hyperbolic measure. The required transformations are described by Bolinder[6] and Deschamps.[7] The transformation of impedances through lossless networks has been treated by Bolinder.[6] By analogy, the transformation of reflection coefficients through the "equivalent lossless" network is identical. Of course, it may be found by the user that the extended method (using the Cayley-Klein Diagram) is simpler. This may be true if Deschamps'[7] hyperbolic protractor is available.

Example

The symmetrical resistive network (reflection coefficient transformer) shown will be considered in an illustration of the method (see Fig. 1). The resistance values were chosen to locate the fixed points (iterative impedances) at ± 1 on the unit circle. If this were not the case, the impedance level could be adjusted. The input impedance is found to be:

$$Z' = \frac{\sqrt{2}Z + 1}{Z + \sqrt{2}}.$$

If this is compared with relation (1) of the Appendix, an analogous transformation,

$$\Gamma' = \frac{\sqrt{2}\Gamma + 1}{\Gamma + \sqrt{2}},$$

may be written if

$$S_{11} = \frac{1}{\sqrt{2}} \qquad S_{12} = \frac{1}{\sqrt{2}} \qquad S_{22} = -\frac{1}{\sqrt{2}}.$$

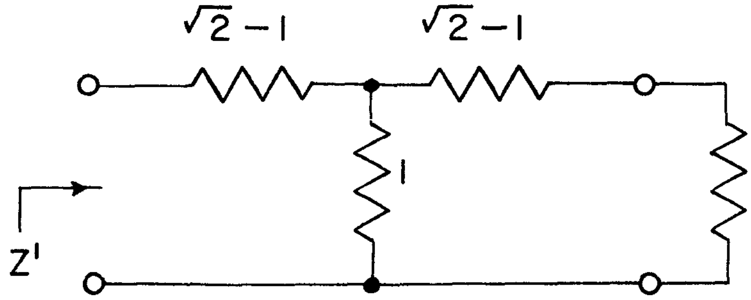

Fig. 1—Reflection coefficient transformer with arbitrary load.

It is noted that the analogous network is lossless, since its scattering matrix is unitary. While the original network is a transformer in the Γ plane, the analogous network is a transformer in the Z plane, whose Z matrix does not exist. Its A, B, C, D matrix is

$$\begin{pmatrix} \sqrt{2}+1 & 0 \\ 0 & \sqrt{2}-1 \end{pmatrix}.$$

Appendix

In general,

$$\Gamma' = \frac{\dfrac{(S_{12}^2 - S_{11}S_{22})\Gamma}{S_{12}} + \dfrac{S_{11}}{S_{12}}}{-\dfrac{S_{22}\Gamma}{S_{12}} + \dfrac{1}{S_{12}}} = \frac{a\Gamma + b}{c\Gamma + d}$$

$$(ad - bc) = 1$$

where

$$S_{11} = |S_{11}| e^{j\Phi_{11}}$$
$$S_{12} = |S_{12}| e^{j\Phi_{12}}$$
$$S_{22} = |S_{22}| e^{j\Phi_{22}}.$$

Now, if we move reference planes 1 and 2 by electrical distances θ_1 and θ_2 along lossless lines, respectively,[8]

$$S_{11}' = |S_{11}| e^{j(\Phi_{11}+2\theta_1)}$$
$$S_{12}' = |S_{12}| e^{j(\Phi_{12}+\theta_1+\theta_2)}$$
$$S_{22}' = |S_{22}| e^{j(\Phi_{22}+2\theta_2)}.$$

Let $\psi = \theta_1 + \theta_2$ (an electrical length of transmission line), and compute $a+d$

$$a + d = \frac{|S_{12}|^2 e^{j(2\Phi_{12}+2\psi)} - |S_{11}S_{22}| e^{j(\Phi_{11}+\Phi_{22}+2\psi)} + 1}{|S_{12}| e^{j(\Phi_{12}+\psi)}}.$$

For the transformation to be nonloxodromic, $a+d$ has to be real. Setting the imaginary part to zero, the result is

$$\tan\psi = \frac{(1 - |S_{12}|^2)\sin\Phi_{12} + |S_{11}S_{22}|\sin(\Phi_{11} + \Phi_{22} - \Phi_{12})}{(|S_{12}|^2 - 1)\cos\Phi_{12} - |S_{11}S_{22}|\cos(\Phi_{11} + \Phi_{22} - \Phi_{12})}.$$

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[5] There is an exception when either k or K is unity, but this is lossless uniform line, which is a trivial case.

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